Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 300 (2007) 265–279
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Constrained generic substructure transformations in finite element model updating

M.J. Terrell, M.I. Friswell*, N.A.J. Lieven

Department of Aerospace Engineering, Queen's Building, University Walk, University of Bristol, Bristol BS8 1TR, UK

Received 13 April 2006; received in revised form 28 July 2006; accepted 8 August 2006
Available online 7 November 2006

Abstract

Model updating is a powerful technique to improve finite element models of structures using measured data. One of the key requirements of updating is a set of candidate parameters that is able to correct the underlying error in the model. Often regions such as joints are very difficult to parameterise satisfactorily using physical design variables such as stiffnesses or dimensions. Parameters arising from generic element and substructure transformations are able to increase the range of candidate parameters, and furthermore are able to correct structural errors. However, unconstrained generic substructure transformations change the connectivity of the model matrices. In many instances retaining the connectivity is desirable and this paper derives constraint equations to do so. The method assumes that substructure eigenvalues are the parameters used in the global updating procedure and that the substructure eigenvector matrix is optimised to enforce the connectivity constraints. The method is demonstrated on a simple L shape test structure, where the substructure is the corner.

1. Introduction

Finite element modelling is a well-established tool to analyse the vibration of structures. Discrepancies often arise between the model and the structure. In many cases these may be corrected by updating design variable parameters, and these techniques are well established [1,2]. In some instances the errors may be caused by the break down in the assumptions underpinning the finite element formulation. A joint in a truss structure modelled using beam elements is a good example. Around the joints the assumed strain distribution model is incorrect leading to a structural error in the model. Ibrahim and Pettit [3] considered the local stiffening due to preload. In order to reduce these structural errors the assumed shape function of the elements must be changed, and this cannot be done by modifying design variable parameters. Additional parameters may be incorporated into the finite element formulation to account for those behavioural affects neglected in the baseline model. More advanced elements can be used, for example Euler–Bernoulli beam elements can be replaced with Timoshenko beam elements to include a parameter that models shear strain. Abstract parameters may be introduced that account for the unmodelled behaviour in an equivalent way. An example is

*Corresponding author. Tel.: +44 117 928 8695.
E-mail address: m.i.friswell@bristol.ac.uk (M.I. Friswell).

0022-460X/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2006.08.026

a rigid beam offset that assumes a portion of the beam near to a joint is rigid. Mottershead et al. [4] used rigid offsets to update an aluminium space frame, and Horton et al. [5] demonstrated their benefits over design variable parameters in the updating of welded joints. In the latter case the resultant offset was negative, implying that the joint was much more flexible than the baseline model.

Stavrinidis et al. [6,7] assumed that the element stiffness matrix for a beam was accurate and derived an optimum element mass matrix for a beam. The mass matrix was obtained by minimising the discretisation error using a Taylor series expansion, while retaining the inertia properties of the rigid body modes. An alternative view of this approach is that it allows a wider choice of shape functions. The method may also be applied to plate elements [8]. This more general approach to parameterisation was extended by Ahmadian et al. [9] and Gladwell and Ahmadian [10] with the introduction of the generic element concept. A generic element is a member of a family of elements that encompasses all possible element formulations described by the same displacement field. Once a member of the family is defined (the baseline element), it is possible to obtain any other element in the family, by modifying the baseline eigenvalues and transforming the baseline eigenvectors. For example an Euler–Bernoulli beam element may be transformed into a Timoshenko beam element. Many additional behavioural affects can be incorporated into a model without explicit parameterisation during the element formulation.

Ahmadian et al. [11] compared the generic element parameterisation with direct methods and the design variable parameterisation using a frame structure. They highlighted the loss of physical meaning with direct methods and the restrictions on the type of changes that can be made with design variable parameters. The generic elements had 140 parameters spread between 28 elements and resulted in a massively under-determined estimation problem. Thus multiple solutions exist that reproduce the test data, making it hard to justify the changes made physically. A least-squares solution was obtained, although it was suggested that additional constraints should be applied to reduce the size of the parameter set. Ratcliffe et al. [12] applied the generic element approach to joint identification using FRF data. The number of generic parameters was limited by enforcing element symmetry and preventing coupling of uncoupled modes. Proportional damping was assumed and the updated model showed significant improvement in simulated and experimental examples. Titurus et al. [13] used the first two eigenvalues of a T-joint substructure as parameters to update a welded frame. A high level of correlation was achieved with regard to both natural frequencies and mode shapes. Friswell et al. [14] used generic elements to improve the dynamic models of golf clubs. The club shaft was modelled using a generic beam element and the parameters included an element eigenvalue. Wu and Law [15,16] highlighted the difference between parameter and structural errors. A model was defined as containing structural errors if significant physical effects had been neglected, and needed to be incorporated during updating. It was highlighted that modifying physical parameters cannot correct these errors as the assumed shape functions would not change. The generic parameters were updated using a flexibility sensitivity method, rather than standard frequency or modeshape sensitivity. Updating showed good correction of the structural errors that would not be possible with design variable parameters. Law et al. [17] modelled and updated a steel space frame. Generic elements were connected by semi-rigid spring joint models and were updated simultaneously, improving the frequency and modeshape correlation with the test data. Generic elements have also been applied to the field of damage detection. Wang et al. [18] located damage in T and L-shaped joints using FRF data.

A large number of potential parameters are available with generic elements or substructures. For example, an in-plane beam element has 3, 4 or 6 parameters if the deformation is considered symmetric and uncoupled (axial and bending deflection), non-symmetric and uncoupled, or non-symmetric and coupled, respectively. If all of the generic element parameters are used then the updating equation will be under-determined, and regularisation techniques will be required to obtain a unique solution. Often a minimum norm solution is used, but this may not be the best option to produce a physically representative model. Titurus et al. [13] applied the generic element approach to substructures to improve stiffness models around the joints, and subset selection was used to reduce the number of parameters.

Substructuring allows the areas of uncertainty to be updated as a region and will incorporate interactions between the elements from which it is composed. However unconstrained substructure transformations will change the connectivity of the substructure because unconnected and uncoupled degree of freedom pairs are active in the same substructure mode. The purpose of this paper is to develop a method to generate constraint equations to ensure the connectivity properties of the baseline substructure are retained. The main application is when the substructure eigenvalues are used as updating parameters and the substructure eigenvector transformation is used to satisfy the connectivity constraints. This significantly reduces the size of the parameter set, while allowing controllable changes to the model matrices and retaining the connectivity of the baseline model.

2. Theory of the generic element method

There are two approaches to the generic element or substructure method. In the following the method will be applied to a substructure, but the method is equally applicable to a single finite element. The first approach uses the solution to the generalised eigenvalue problem corresponding to the substructure mass and stiffness matrices to make changes to both the mass and stiffness. The second approach only changes the stiffness using the eigendata of the substructure stiffness matrix. The mass matrix may also be changed in isolation, although usually the mass distribution is more accurately modelled. Gladwell and Ahmadian [10] gave a full derivation of the generic element theory.

2.1. Combined mass and stiffness substructure decomposition

A baseline substructure is defined from which the transformations will take place. Baseline matrices are identified by the subscript 0 and the matrices corresponding to the substructure are denoted by the superscript s. The eigenvalue problem is given by

$$\mathbf{K}_0^s\mathbf{\Phi}_0^s = \mathbf{M}_0^s\mathbf{\Phi}_0^s\mathbf{\Lambda}_0^s, \tag{1}$$

where $\mathbf{\Phi}_0^s$ is the mass normalised eigenvector matrix, $\mathbf{M}_0^s$ and $\mathbf{K}_0^s$ are the mass and stiffness matrices of the substructure and $\mathbf{\Lambda}_0^s$ is the diagonal matrix of eigenvalues. Assuming the eigenvectors are mass normalised, $\mathbf{\Phi}_0^s$ and $\mathbf{\Lambda}_0^s$ satisfy

$$\mathbf{\Phi}_0^{s\mathrm{T}}\mathbf{M}_0^s\mathbf{\Phi}_0^s = \mathbf{I} \quad \text{and} \quad \mathbf{\Phi}_0^{s\mathrm{T}}\mathbf{K}_0^s\mathbf{\Phi}_0^s = \mathbf{\Lambda}_0^s. \tag{2}$$

The transformed substructure matrices are identified by the subscript T. The eigenvectors of the baseline and transformed substructure are related by the transformation matrix $\mathbf{S}$, where

$$\mathbf{\Phi}_0^s = \mathbf{\Phi}_T^s\mathbf{S}. \tag{3}$$

The baseline and transformed substructures are assumed to have the same number of rigid body modes; d. The format of $\mathbf{S}$ ensures that the baseline rigid body modes are linear combinations of the transformed rigid body modes only; whereas the baseline strain modes are a linear combination of both transformed rigid body and strain modes. Thus,

$$[\mathbf{\Phi}_{0R}^s \;\; \mathbf{\Phi}_{0S}^s] = [\mathbf{\Phi}_{TR}^s \;\; \mathbf{\Phi}_{TS}^s]\begin{bmatrix} \mathbf{S}_R^s & \mathbf{S}_{RS}^s \\ \mathbf{0} & \mathbf{S}_S^s \end{bmatrix}, \tag{4}$$

where the subscripts R and S denote the rigid body and strain modes, respectively. The eigenvalue problem for the transformed substructure is

$$\mathbf{K}_T^s\mathbf{\Phi}_T^s = \mathbf{M}_T^s\mathbf{\Phi}_T^s\mathbf{\Lambda}_T^s. \tag{5}$$

Assuming the eigenvectors are mass normalised, the modified substructure eigenvalue matrix, $\mathbf{\Lambda}_T$, and the transformed substructure eigenvector matrix, $\mathbf{\Phi}_T$, satisfy

$$\mathbf{\Phi}_T^{s\mathrm{T}}\mathbf{M}_T^s\mathbf{\Phi}_T^s = \mathbf{I} \quad \text{and} \quad \mathbf{\Phi}_T^{s\mathrm{T}}\mathbf{K}_T^s\mathbf{\Phi}_T^s = \mathbf{\Lambda}_T^s. \tag{6}$$

Eqs. (2)–(6) can be rearranged to give

$$\mathbf{M}_T^s = \mathbf{M}_0^s\mathbf{\Phi}_0^s\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{\Phi}_0^{s\mathrm{T}}\mathbf{M}_0^s \tag{7}$$

and

$$\mathbf{K}_T^s = \mathbf{M}_0^s \mathbf{\Phi}_{0S}^s \mathbf{S}_S^{\mathrm{T}} \mathbf{\Lambda}_{TS}^s \mathbf{S}_S \mathbf{\Phi}_{0S}^{s\mathrm{T}} \mathbf{M}_0^s \tag{8}$$

where $\mathbf{\Lambda}_{TS}^s$ is the diagonal matrix of strain eigenvalues for the transformed substructure.

Symmetric matrices $\mathbf{U}$ and $\mathbf{V}$ may be defined that describe the transformation of the substructure mass and stiffness matrices, respectively, and are

$$\mathbf{U} = \mathbf{S}^{\mathrm{T}}\mathbf{S} \quad \text{and} \quad \mathbf{V} = \mathbf{S}_S^{\mathrm{T}} \mathbf{\Lambda}_T^s \mathbf{S}_S. \tag{9}$$

2.2. Transforming the substructure stiffness matrix only

Essentially the procedure is equivalent to that described in the previous section, where the mass matrix is set to the identity. If $\mathbf{\Psi}_0^s$ denotes the eigenvector matrix of the substructure stiffness matrix, and $\mathbf{\Sigma}_0^s$ is the corresponding diagonal matrix of eigenvalues, then

$$\mathbf{K}_0^s = \mathbf{\Psi}_0^s \mathbf{\Sigma}_0^s \mathbf{\Psi}_0^{s\mathrm{T}}. \tag{10}$$

The modified eigenvalues are given by $\mathbf{\Sigma}_T$, and the transformed eigenvectors are related to the baseline eigenvectors by the matrix $\mathbf{R}$, where

$$\mathbf{\Psi}_0^s = \mathbf{\Psi}_T^s \mathbf{R}. \tag{11}$$

The transformed stiffness matrix is then

$$\mathbf{K}_T^s = \mathbf{\Psi}_0^s \mathbf{R}^{\mathrm{T}} \mathbf{\Sigma}_T^s \mathbf{R} \mathbf{\Psi}_0^{s\mathrm{T}}. \tag{12}$$

However, there is an additional constraint on the transformation matrix. In the method described in Section 2.1 the mass matrix was able to change. Here the normalisation of the eigenvectors requires that

$$\mathbf{\Psi}_0^{s\mathrm{T}} \mathbf{\Psi}_0^s = \mathbf{I} \quad \text{and} \quad \mathbf{\Psi}_T^{s\mathrm{T}} \mathbf{\Psi}_T^s = \mathbf{I}. \tag{13}$$

Substituting Eq. (11) into Eq. (13) yields,

$$\mathbf{R}^{\mathrm{T}} \mathbf{\Psi}_T^{s\mathrm{T}} \mathbf{\Psi}_T^s \mathbf{R} = \mathbf{I} \quad \Rightarrow \quad \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}. \tag{14}$$

Thus the transformation $\mathbf{R}$ must be an orthogonal matrix. When using the mass normalised decomposition in Section 2.1 no such restriction on the transformation matrix applies.

Although this section has considered the substructure stiffness matrix in isolation, a similar approach can be taken to transform the mass matrix.

2.3. Separation of mass and stiffness matrix transformations

Consider the baseline structure mass and stiffness matrices, $\mathbf{M}_0$ and $\mathbf{K}_0$. The method of Section 2.1 automatically infers changes in both the mass and stiffness matrices. We can obtain changes in the stiffness matrix only using the method of Section 2.2, but this requires further a constraint on the substructure eigenvector transformations. Satisfying the constraint of an orthogonal $\mathbf{R}$ can be difficult, particularly when considering complex substructure models.

The question arises whether we can use the transformed stiffness matrix from Section 2.1, and simply ignore the changes to the mass matrix. Expressed formally, given a substructure eigenvector transformation matrix $\mathbf{S}$ and transformed eigenvalues $\mathbf{\Lambda}_T$, does there exist a transformation matrix $\mathbf{R}$ and transformed eigenvalue matrix $\mathbf{\Sigma}_T$ such that the stiffness matrices given by Eqs. (8) and (12) are identical. Consideration of the number of parameters and constraints demonstrates that finding such a solution is, in general, impossible. However, when a lumped mass matrix is used, the examples show that a suitable $\mathbf{R}$ and $\mathbf{\Sigma}_T$ may be found.

3. Zero stiffness constraints

When considering a substructure, there will be uncoupled and unconnected degree of freedom pairs. For example suppose a T-shaped substructure is modelled using four nodes, with three nodes at ends of the legs and the fourth node at the joint. If the legs of the structure are sufficiently thin so that a beam model is sufficiently accurate, then the degrees of freedom at one of the outside nodes are not connected directly to any of the degrees of freedom at the other outside nodes. Of course there will be an indirect connection through the central node. However the unconnected degrees of freedom mean that there will be a zero in the stiffness matrix at an off-diagonal position corresponding to an unconnected degree of freedom pair. If the analyst is convinced that these unconnected degrees of freedom should remain unconnected then this zero in the stiffness matrix should be retained.

Levin et al. [19] showed that early direct methods which altered connectivity properties were not physically meaningful. In general the generic substructure method does not retain the connectivity of the baseline model. In the case of uncoupled degrees of freedom, a substructure with additional non-zero elements in $\mathbf{K}_T$ will exhibit a new type of behaviour. Such a change in the structure of the model matrices may be beneficial when the basis of the substructure model is uncertain. However controlling the connectivity of the substructure degrees of freedom can have significant benefits, and the rest of this paper will consider methods that retain the connectivity. Only changes to the stiffness matrix are considered further, and hence only strain modes are required. Thus the subscripts R and S are dropped. A lumped substructure mass matrix is used for the development.

Each element of the stiffness matrix is made up of a contribution from each mode. Element (i,j) of the baseline substructure stiffness matrix can be expressed as

$$\mathbf{K}^s_{0ij} = \mathbf{M}^s_{0ii}\mathbf{M}^s_{0jj}\sum_{p=1}^{m}\mathbf{\Lambda}^s_{0pp}\mathbf{\Phi}^s_{0ip}\mathbf{\Phi}^s_{0jp}, \tag{15}$$

where $\mathbf{\Phi}^s_{0ip}$ is the ith element of mode p of the baseline substructure eigenvector matrix (strain modes only) and m is the number of strain modes. For each value of the summation index, the contribution of a different mode is identified. Suppose only the substructure eigenvalues are modified ($\mathbf{S} = \mathbf{I}$). Then,

$$\mathbf{K}^s_{Tij} = \mathbf{M}^s_{0ii}\mathbf{M}^s_{0jj}\sum_{p=1}^{m}\mathbf{\Lambda}^s_{Tpp}\mathbf{\Phi}^s_{0ip}\mathbf{\Phi}^s_{0jp}. \tag{16}$$

Inspection of Eqs. (15) and (16) shows that a transformation of eigenvalue p will perturb stiffness term $\mathbf{K}_{0ij}$ if degrees of freedom i and j are both active in mode p. An eigenvalue perturbation will therefore change the connectivity of the baseline model.

If the zeros in the stiffness matrix are to be preserved then

$$\mathbf{K}^s_{Tij} = 0 \quad \text{when} \quad \mathbf{K}^s_{0ij} = 0, \tag{17}$$

where

$$\mathbf{K}^s_{Tij} = \mathbf{M}^s_{0ii}\mathbf{M}^s_{0jj}\sum_{p=1}^{m}\mathbf{\Lambda}^s_{Tpp}\mathbf{\Phi}^s_{Iip}\mathbf{\Phi}^s_{Ijp} \tag{18}$$

and

$$\mathbf{\Phi}^s_I = \mathbf{\Phi}^{s\mathrm{T}}_0\mathbf{S}^s. \tag{19}$$

The zero stiffness constraints are identified using Eq. (17). The approach adopted here will be to use the elements of the transformed eigenvalues $\mathbf{\Lambda}^s_T$ as the updating parameters, and then optimise the eigenvector transformation matrix $\mathbf{S}$ to retain the connectivity. The optimisation is nonlinear, with the associated uniqueness and existence questions, and cannot be solved analytically. Rather than find an exact solution, we minimise the residual in the zero stiffness constraint equations, using an error function of the form

$$E(\mathbf{S}) = \sum_{(i,j)\in Z}\|\mathbf{K}^s_{Tij}\|^2 \tag{20}$$

where Z is the set matrix elements corresponding to the zeros in the baseline substructure stiffness matrix. The changes in the substructure eigenvector matrix should be small and a good initial estimate of $\mathbf{S}$ is the identity matrix.

A constraint equation is defined for each zero that is to be preserved in $\mathbf{K}_T$ using Eq. (18). The variables in the optimisation are the elements of $\mathbf{S}$. If r is the number of rigid body modes, and m the number of flexible modes, then the number of variables (n_{var}) and the number of constraints (n_{con}) are

$$n_{\text{var}} = m^2 \quad \text{and} \quad n_{\text{con}} < (m+r)(m+r-1)/2. \tag{21}$$

A substructure will be composed of two or more elements. In the case of beam elements with bending and axial motion in one plane,

$$m \geqslant 3r \quad \Rightarrow \quad n_{\text{var}} > n_{\text{con}}. \tag{22}$$

The large number of variables in $\mathbf{S}$ and the relatively small number of zero constraints means that the solution will not be unique. One approach is to minimise the change in the substructure eigenvectors, and this is easily included in the optimisation function, Eq. (20). Hence the function to be optimised becomes,

$$E(\mathbf{S}) = \sum_{(i,j)\in Z} \|\mathbf{K}^s_{Tij}\|^2 + \alpha\|\mathbf{S}-\mathbf{I}\|^2 \tag{23}$$

where α is the regularisation parameter. This approach is similar to Tikhonov regularisation [20], although the criterion for choosing α is different. Usually α is chosen to balance the residual and the parameter change, through techniques such as the L-curve [21]. However here the objective is to make the residual small, and the second term in Eq. (23) is merely to ensure a unique solution to the optimisation problem. Hence α should be chosen such that the solution is stable and the residual is sufficiently small. This will be demonstrated in the example.

Suppose eigenvalue p is modified, so that $\mathbf{\Lambda}^s_{Tpp} = \mu_p \mathbf{\Lambda}^s_{0pp}$. There will be tendancy for the optimisation to scale $\mathbf{S}_{pp}$ by $1/\sqrt{\mu_p}$ to absorb the perturbations in the constraint equations and the trivial solution given by

$$\mathbf{S}^{\mathrm{T}}\mathbf{\Lambda}^s_T\mathbf{S} = \mathbf{\Lambda}^s_0 \tag{24}$$

will be obtained. The diagonal element of $\mathbf{S}$ corresponding to the eigenvalue that is being modified is therefore set equal to 1.

A trust region method utilising the Matlab 'fsolve' function is used to solve the optimisation problem. The Jacobian can be derived analytically which greatly increases the efficiency of the optimisation. Element (g,h) of the Jacobian is the partial derivative of constraint g with respect to variable h. Each constraint is based upon an element of the stiffness matrix. It is assumed that constraint g corresponds to element (i,j) of the stiffness matrix and that variable h corresponds to element (q,r) of the transformation matrix. A lumped mass matrix is assumed. Each row of $\mathbf{S}$ corresponds to a particular mode, and hence variable h will affect mode q only. The constraints defined by Eq. (18) can be broken down into a contribution from each mode. The only term in the summation that is sensitive to variable h is when $p = q$. Thus,

$$\mathbf{J}_{gh} = \frac{\partial \mathbf{K}^s_{Tij}}{\partial \mathbf{S}_{qr}} = \mathbf{M}^s_{0ii}\mathbf{M}^s_{0jj}\frac{\partial(\mathbf{\Lambda}^s_{0qq}\mathbf{\Phi}^s_{Iiq}\mathbf{\Phi}^s_{Ijq})}{\partial \mathbf{S}_{qr}}. \tag{25}$$

$\mathbf{\Phi}^s_I$ is expanded to include $\mathbf{S}$ explicitly as

$$\mathbf{\Phi}^s_{Iiq} = \mathbf{\Phi}^s_{0i:}\mathbf{S}^{\mathrm{T}}_{q:}, \tag{26}$$

where the notation $\mathbf{A}_{i:}$ and $\mathbf{A}_{:i}$ indicates row i and column i of the matrix $\mathbf{A}$, respectively. Therefore,

$$\mathbf{J}_{gh} = \mathbf{M}^s_{0ii}\mathbf{M}^s_{0jj}\mathbf{\Lambda}^s_{0qq}\frac{\partial(\mathbf{\Phi}^s_{0i:}\mathbf{S}^{\mathrm{T}}_{q:}\mathbf{\Phi}^s_{0j:}\mathbf{S}^{\mathrm{T}}_{q:})}{\partial \mathbf{S}qr} \tag{27}$$

where

$$\mathbf{\Phi}^s_{0i:}\mathbf{S}^{\mathrm{T}}_{q:} = \sum_p^m \mathbf{\Phi}^s_{0ip}\mathbf{S}_{qp}. \tag{28}$$

The only term in the summation that is sensitive to variable h is when $p=r$. Using the chain rule to differentiate the product in Eq. (27) yields

$$\mathbf{J}_{gh} = \mathbf{M}^s_{0ii}\mathbf{M}^s_{0jj}\mathbf{\Lambda}^s_{0qq}(\mathbf{\Phi}^s_{0ir}\mathbf{\Phi}^s_{0j:}\mathbf{S}^{\mathrm{T}}_{q:} + \mathbf{\Phi}^s_{0jr}\mathbf{\Phi}^s_{0i:}\mathbf{S}^{\mathrm{T}}_{q:}). \tag{29}$$

It is worth commenting at this stage on the assumptions made regarding the properties of the substructures considered. A lumped mass matrix is used because it simplifies Eqs. (15) and (16) resulting in the mass terms appearing as scalar operators outside of the summation. Alternative mass matrices can be used to derive equivalent equations. When using generic substructures to model real structural components an accurate mass model is required, and the lumped mass model is rarely sufficient. As discussed in Section 2.3, the stiffness matrix will be modified based on the transformation derived using the lumped mass matrix. However, the global eigenvalues are computed using this modified stiffness matrix with a consistent mass matrix. This means that the simplicity in the equations can be maintained without compromising the mass model.

4. Case study

In order to update a model using the method presented, the substructure must first be isolated from the global structure. The level of perturbation of the substructure eigenvalue is defined by the eigenvalue ratio μ_p where,

$$\mu_p = \frac{\mathbf{\Lambda}^s_{Tpp}}{\mathbf{\Lambda}^s_{0pp}}. \tag{30}$$

The constraint equations are identified and solved numerically for each substructure mode over a range of μ_p. This gives the variation of $\mathbf{K}^s_T$ with respect to μ_p, and hence μ_p will be used as the updating parameter. The data set containing the variation of $\mathbf{K}^s_T$ with respect to μ_p is saved for each substructure mode so that at each iteration in the updating scheme the relevant data can be accessed. Linear interpolation is used between data points, and finite differences are used to calculate sensitivities. Fig. 1 shows a flow chart of the procedure.

Fig. 2 shows the test structure and corresponding finite element model. The baseline model is constructed using Euler–Bernoulli beam elements with a consistent mass model. The distinctive fillet radius at the corner will be treated as the area of uncertainty and will be modelled using two beam elements of length 40 mm to form a corner substructure (using a lumped mass model to calculate the eigenvector transformations). The substructure modeshapes are shown in Fig. 3. The substructure has an axis of symmetry which is preserved through the transformations by preventing modes with different symmetry properties from mixing. Fig. 4 identifies the symmetric and anti-symmetric modes of the corner substructure. The representation of the mode shapes in Fig. 4 shows only the sign of the degree of freedom within each mode. For example if degree of freedom pair 3–9 is examined it is clear that they are symmetric in modes 1, 3 and 5 and anti-symmetric in modes 2, 4 and 6. When selecting parameters for the optimisation, only those elements of the eigenvector transformation matrix that combine modes with the same symmetry properties are required, and these are identified by a 'tick' in the representation of **S** given.

All six substructure eigenvalues are perturbed over the range $0.5<\mu_p<2$. The regularisation parameter α is determined using Figs. 5(a)–(f) which show plots of the residual and the side constraint for $p=1,\ldots,6$ where $\mu_p=1.5$. For $\alpha<10^{-4}$ the solution is seen to be stable. Figs. 6–8 shows the variation in the elements of the substructure stiffness matrix over the range of the generic parameters. The perturbation of eigenvalues 1 and 2 have almost identical effects on the stiffness matrix and so only one of these eigenvalues needs to be considered (eigenvalue 2 is selected). Perturbation of eigenvalues 5 and 6 introduce changes to axial degrees of freedom only and can therefore be discounted.

The structure was tested under free–free conditions using roving hammer excitation and a fixed accelerometer. The measured and analytical global natural frequencies are shown in Table 1 and Fig. 9 shows the global modeshapes. The first seven flexible modes will be used; the first five will be updated using the sensitivity method, and the final two will be used to verify whether the changes made represent an overall

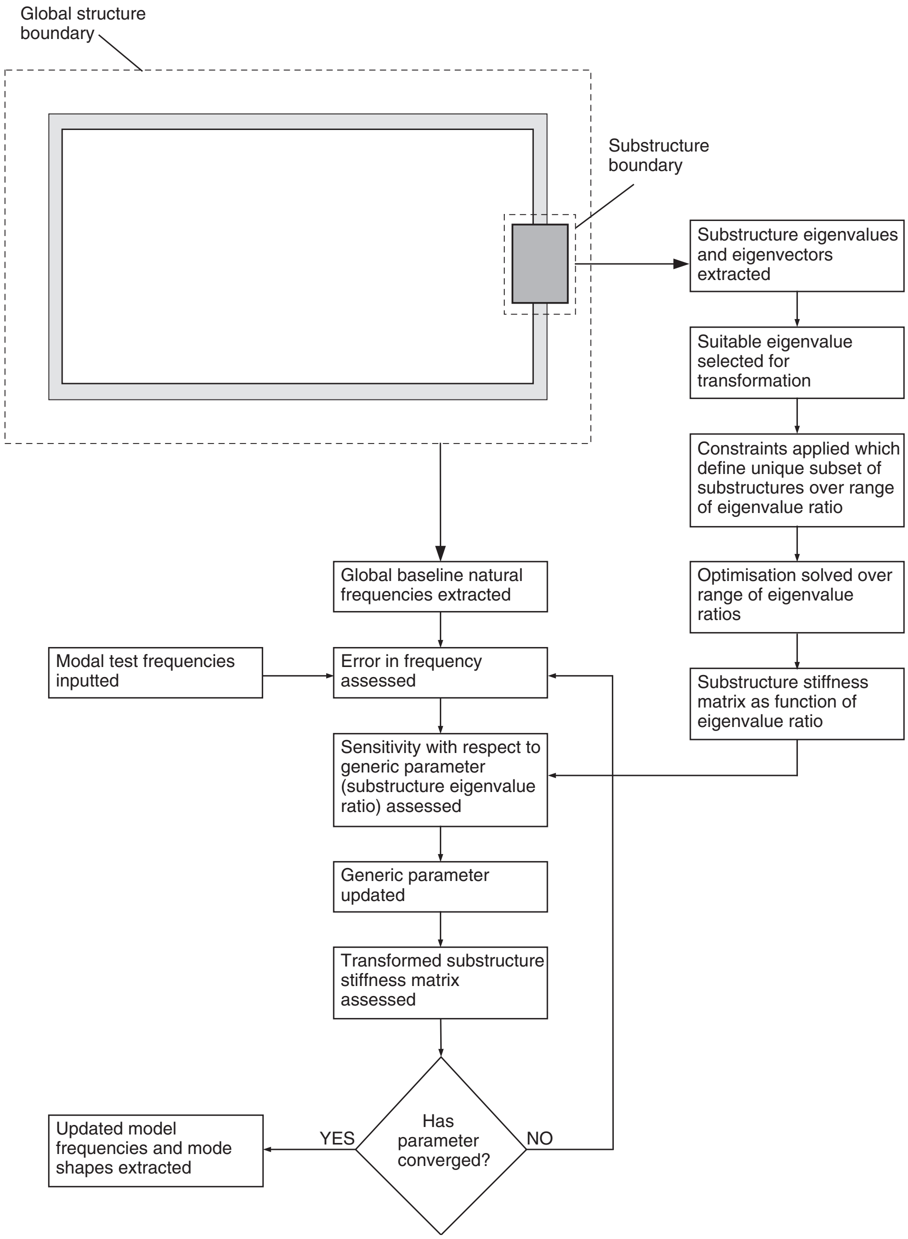

Fig. 1. Flow chart detailing the updating procedure using constrained generic substructures.

improvement in the model. A sensitivity study of the global natural frequencies with respect to the potential updating parameters is shown in Table 2. The parameters are Global Young's modulus, E_G, the substructure Young's Modulus, E_S, and three substructure eigenvalues.

Four updating schemes will be considered. Each scheme will include two parameters. Global Young's modulus, E_G, will be used in each case (to incorporate global stiffening) along with one substructure parameter. This will be substructure Young's Modulus, E_S, or one of the remaining substructure parameters, μ_2, μ_3 and μ_4. Global natural frequency data is used for the updating, and each natural frequency is weighted by the corresponding measured natural frequency. In each scheme the sensitivity matrix is well conditioned, the updating equation is over-determined and regularisation is not required. The Young's modulus terms were

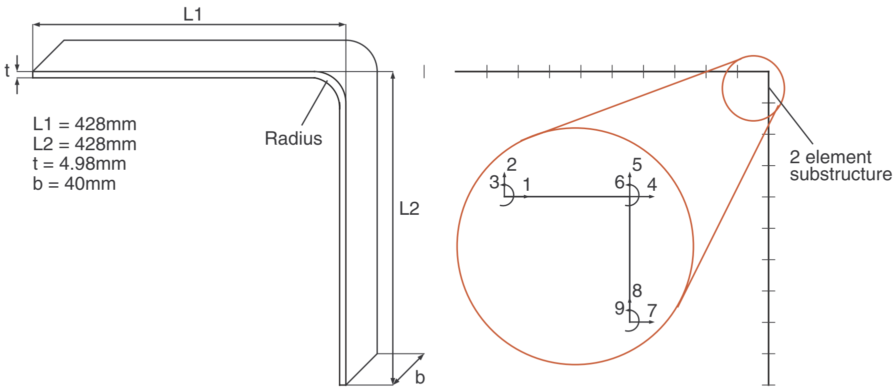

Fig. 2. The example structure and global finite element model.

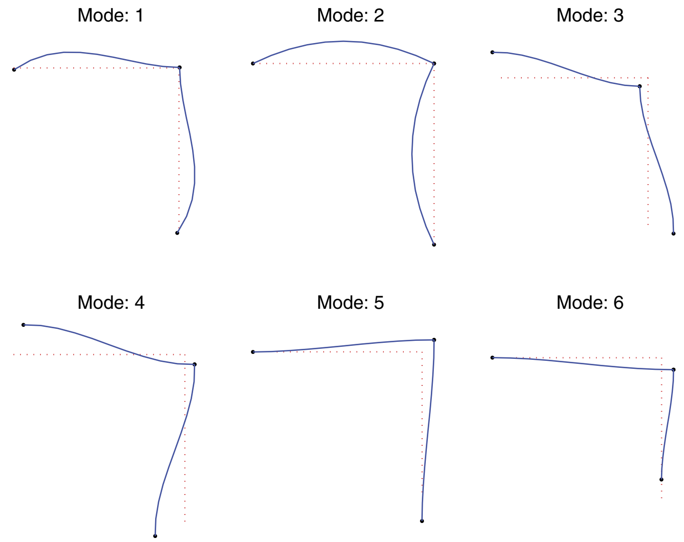

Fig. 3. Corner substructure strain modeshapes.

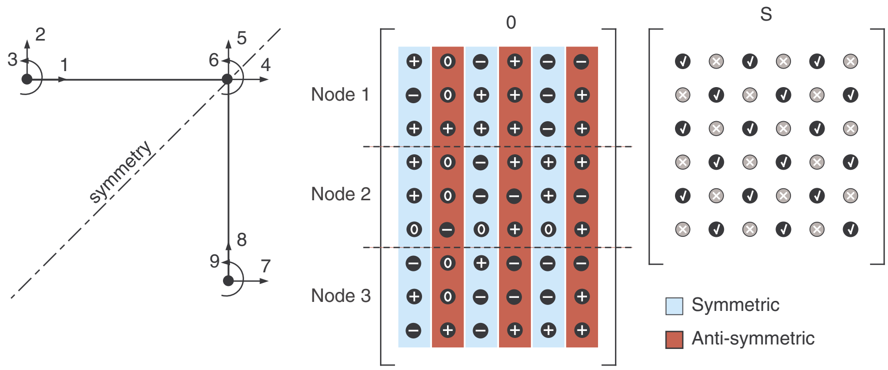

Fig. 4. The symmetry properties of the corner substructure.

updated as a ratio of the initial value. No limits were placed on the maximum change to these parameters. The generic substructure parameters were bounded within the range of the transformation applied; $0.5<\mu_p<2$. Tables 3 and 4 show the results of the updating schemes and Fig. 10 shows the parameter convergence.

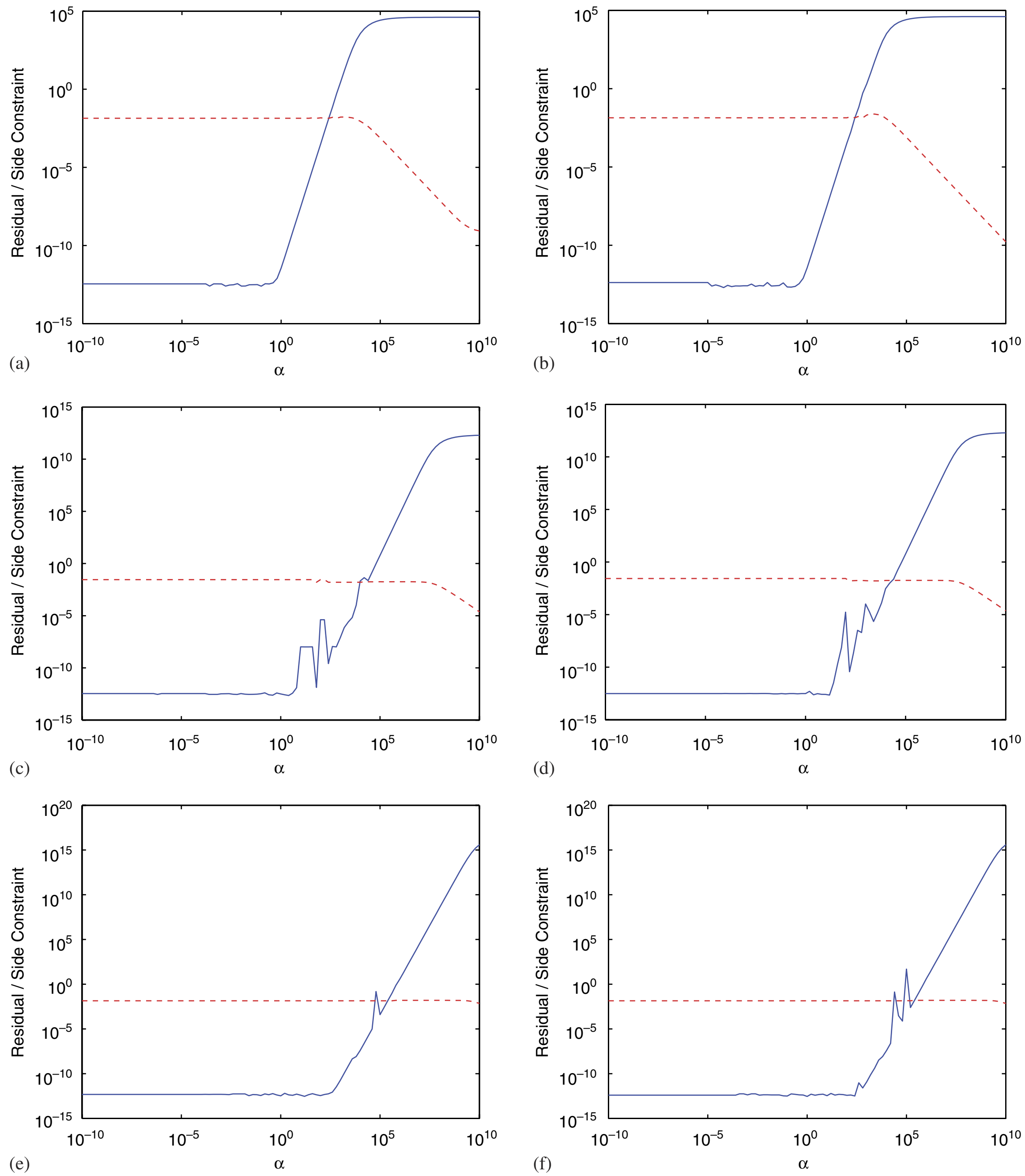

Fig. 5. The variation in the residual and side constraint with the regularisation parameter (α) for corner substructure eigenvalue perturbations given by $\mu_p = 1.5$, where (a)–(f) correspond to eigenvalues 1–6, respectively. Residual, side constraint.

Table 3 shows that all schemes improved the overall natural frequency correlation significantly, including those frequencies that were omitted during updating. The global Young's modulus changed very little in all schemes which indicates that the initial value was accurate and that the error in the baseline model was concentrated in the selected substructure. Table 4 shows the average magnitude of the error for the five updated modes, and for all seven modes. Young's modulus gives the (marginally) best correlation when only the updated modes are considered. When the two higher modes are included μ_3 and μ_4 give the best overall correlation.

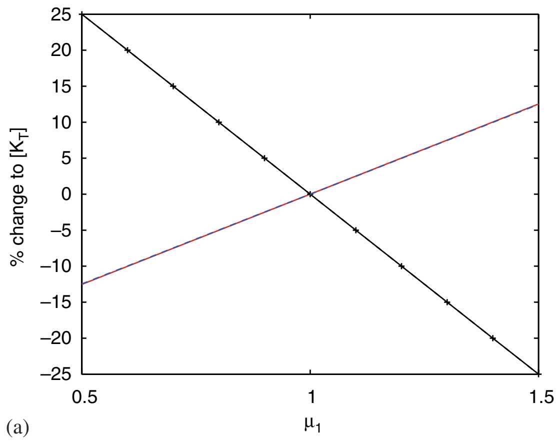

Fig. 6. Stiffness changes obtained by transforming corner substructure eigenvalues 1 and 2: (a) eigenvalue 1, (b) eigenvalue 2. - - - dofs 3-3, - + - dofs 3-6, 6-9, —— dofs 6-6.

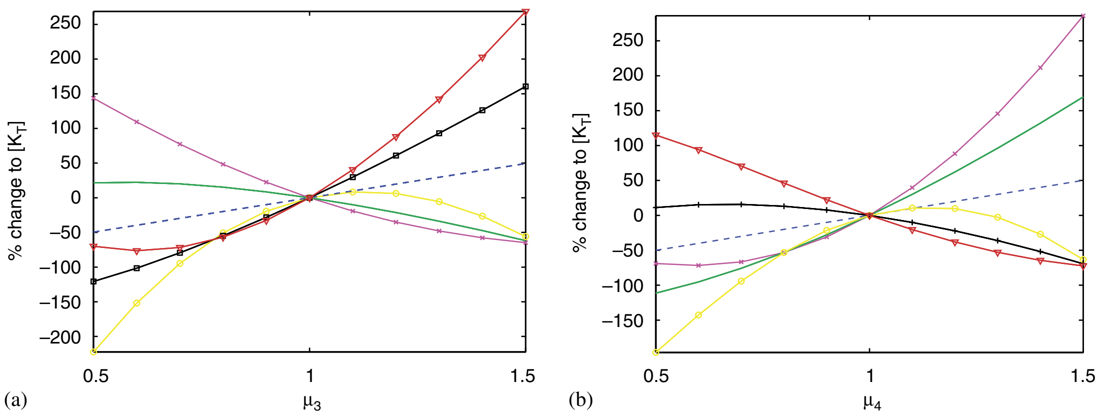

Fig. 7. Stiffness changes obtained by transforming corner substructure eigenvalues 3 and 4: (a) eigenvalue 3, (b) eigenvalue 4. - - - dofs: 2-2, 2-5, 4-7, 7-7, - + - dofs: 2-3, 3-5, 4-9, 7-9, —— dofs: 2-6, 4-6, 5-6, 6-7, - ▽ - dofs: 3-3, 9-9, - ○ - dofs: 3-6, 6-9, - × - dofs: 6-6.

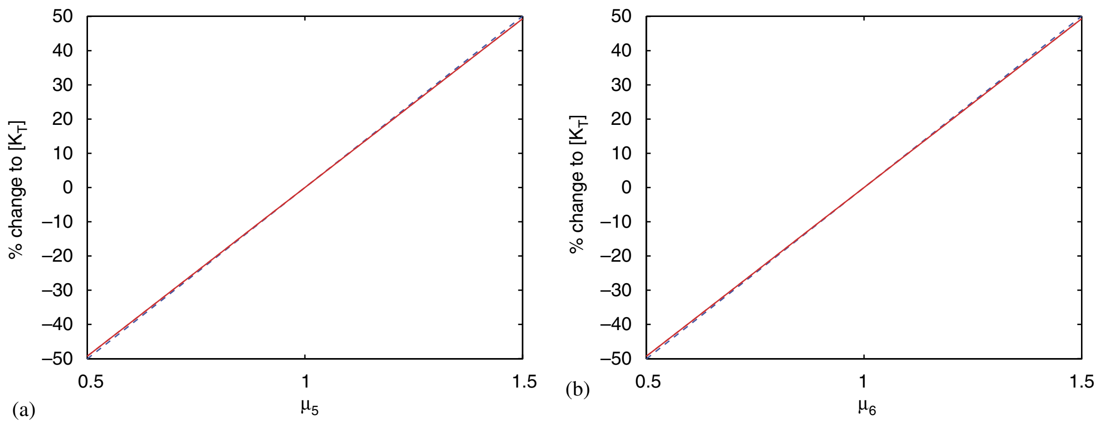

Fig. 8. Stiffness changes obtained by transforming corner substructure eigenvalues 5 and 6: (a) eigenvalues 5, (b) eigenvalue 6. - - - dofs: 1-1, 1-4, 5-8, 8-8, —— dofs: 4-4, 5-5.

Table 1
The global natural frequencies of the example structure

Global mode	Test frequency (Hz)	Model frequency (Hz)	Error (%)
1	28.4	27.5	−3.2
2	106.3	106.4	0.1
3	160.9	154.6	−3.9
4	331.7	331.1	−0.2
5	432.8	414.2	−4.3
6	686.5	684.0	−0.4
7	838.8	802.0	−4.4

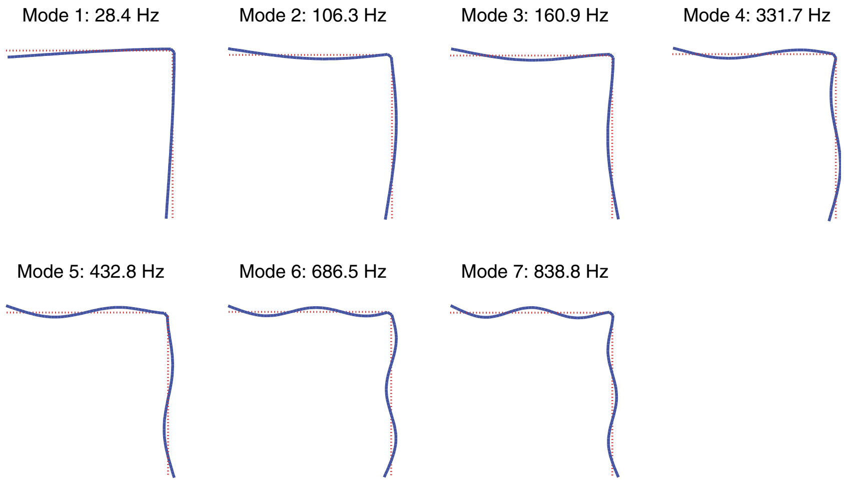

Fig. 9. Global modeshapes and natural frequencies for the structure.

Table 2
The sensitivity of the global modes to the updating parameters

Global mode	E_G	E_S	μ_2	μ_3	μ_4
1	13.7	4.1	4.1	1.5	0.1
2	53.0	0.5	0.4	1.4	−0.5
3	77.1	17.9	17.5	−6.8	7.6
4	165.1	4.3	3.3	13.5	−4.7
5	206.6	35.5	32.5	−45.1	33.9

Table 3
The natural frequencies of the finite element model, before and after updating

Mode	Baseline model (Hz)	E_S (Hz)	μ_2 (Hz)	μ_3 (Hz)	μ_4 (Hz)
1	27.5	28.8	28.8	27.9	27.9
2	106.4	106.5	106.6	107.6	107.4
3	154.6	160.5	160.4	160.4	160.4
4	331.1	332.4	332.5	332.2	332.4
5	414.2	426.5	425.9	435.3	435.5
6	684.0	689.0	688.7	679.0	681.2
7	802.0	820.1	817.0	846.4	850.0

Table 4
Errors in the natural frequencies in the finite element model, before and after updating

Mode	Baseline model (%)	E_S (%)	μ_2 (%)	μ_3 (%)	μ_4 (%)
1	−3.2	+1.3	+1.3	−1.7	−1.6
2	+0.1	+0.2	+0.3	+1.2	+1.1
3	−3.9	−0.3	−0.3	−0.3	−0.3
4	−0.2	+0.2	+0.2	+0.1	+0.2
5	−4.3	−1.4	−1.6	+0.6	+0.6
6	−0.4	+0.4	+0.3	−1.1	−0.8
7	−4.4	−2.2	−2.6	+0.9	+1.3
Average 1–5	2.3	0.7	0.7	0.8	0.8
Average 1–7	2.3	0.9	1.0	0.8	0.8

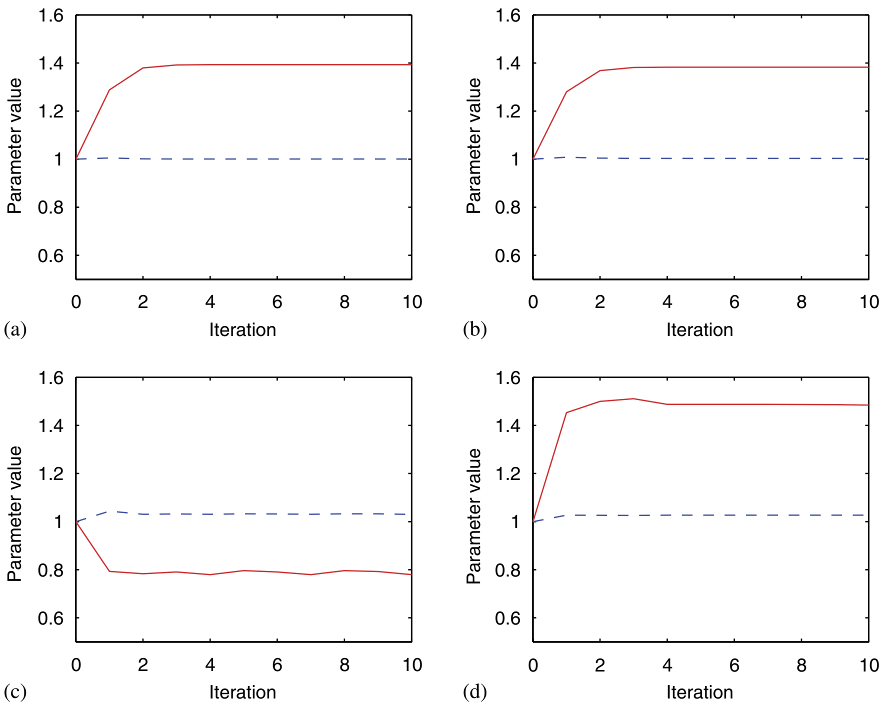

Fig. 10. Convergence of the parameter values during updating. Substructure parameters: (a) E_S, (b) μ_{C2}, (c) μ_{C3} and (d) μ_{C4}. - - - E_G, —— substructure parameter.

Tables 2 and 3 show that the parameters E_S and μ_2 have a very similar effect on the global modes. The best solution will have a high level of correlation but will also introduce minimal changes to the stiffness matrix. The final parameter value in each case is roughly 1.5. When considering E_{sub} this will infer a +50% change to every element of the substructure stiffness matrix. When considering the generic parameter the changes are far more specific. For example when $\mu_2 = 1.5$ the changes in typical elements of the substructure stiffness matrix are:

$$\text{dof } 3-6 \;\Rightarrow\; -25\%,$$

$$\text{dof } 6-6 \;\Rightarrow\; +10\%,$$

$$\text{dof } 6-9 \;\Rightarrow\; -25\%.$$

The changes made using the generic parameter are targeted toward the influential degrees of freedom and represent a far more elegant correction of the baseline matrix than that obtained using substructure Young's modulus.

5. Equivalent generic element transformations

There are two types of constraints placed on the transformations: those between unconnected degree of freedom pairs and those between uncoupled degree of freedom pairs. Preservation of the zeros between unconnected degree of freedom pairs ensures that the constrained substructure transformation can be described by an equivalent set of transformations, applied to each of the elements that make up the substructure. Preservation of the zeros between the uncoupled degree of freedom pairs effectively infers constraints on the form of **S** in the equivalent element transformations. In the case considered here the symmetry of the substructure was preserved so the two elements from which it is composed are identical and a single element transformation will describe the substructure transformation. Coupling constraints were preserved, hence the matrix **V** from Eq. (9) has the form

$$\mathbf{V} = \begin{bmatrix} v_{1,1} & v_{2,1} & 0 \\ v_{2,1} & v_{2,2} & 0 \\ 0 & 0 & v_{3,3} \end{bmatrix}. \tag{31}$$

A constrained substructure transformation is described by a single parameter, μ, but will infer changes on all active elements of **V**, for each element in the substructure. The constrained transformation therefore defines relationships between the parameters that make up the equivalent element transformations. These relationships will be determined by the constraints, the selection of the parameters within the optimisation (a symmetrical parameter set was used here), the eigenvalue selected for perturbation and the level to which it is perturbed. The substructure presented here was composed of two identical beam elements so there are four element parameters which are condensed into a single substructure parameter for each mode examined. The benefit in terms of the parameter set reduction in this case is not significant, but it will become so as the substructure size increases. An in-plane T-joint model composed of 4 beam elements, with an axis of symmetry as shown in Fig. 11 will be described by 12 independent element transformation parameters, which can be reduced to a single parameter. Even when applied to very large substructures the parameter set will remain small, whilst enabling significant and controllable changes to be made to the model matrices.

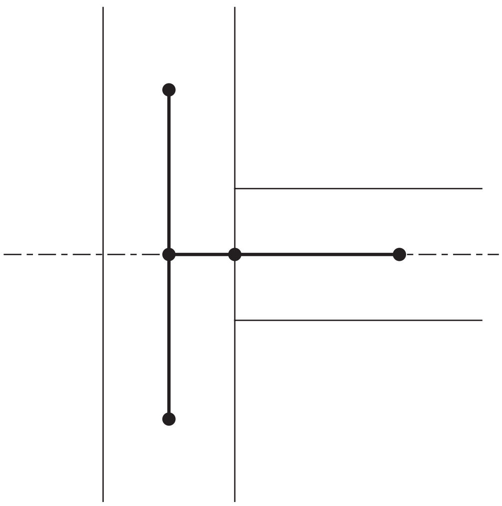

Fig. 11. Proposed T-joint substructure model.

6. Conclusions

A substructure parameterisation technique has been presented which allows changes to be made to the structure of the model matrices whilst preserving the connectivity of the baseline model. The technique requires the numerical solution of an optimisation problem and therefore introduces an extra stage in the updating scheme which increases computational expense. A simple structure was updated to demonstrate the method. This example did not contain significant structural errors and so benefits in terms of the natural frequency correlation over a design variable parameterisation were not apparent. However, it was shown that the updated model using constrained generic substructure parameters represented a more elegant solution.

Acknowledgements

Terrell gratefully acknowledges the support of the Engineering and Physical Sciences Research Council. Friswell gratefully acknowledges the support of a European Commission Marie Curie Excellence Grant and a Royal Society-Wolfson Research Merit Award.

References

[1] J.E. Mottershead, M.I. Friswell, Model updating in structural dynamics: a survey, *Journal of Sound and Vibration* 167 (1993) 347–375.
[2] M.I. Friswell, J.E. Mottershead, *Finite Element Model Updating in Structural Dynamics*, Kluwer Academic Press, Dordrecht, 1995.
[3] R.A. Ibrahim, C.L. Pettit, Uncertainties and dynamic problems of bolted joints and other fasteners, *Journal of Sound and Vibration* 279 (2005) 857–936.
[4] J.E. Mottershead, C. Mares, M.I. Friswell, Selection and updating of parameters for an aluminium space-frame model, *Mechanical Systems and Signal Processing* 14 (2000) 923–944.
[5] B. Horton, H. Gurgenci, M. Veidt, M.I. Friswell, Finite element model updating of welded joints in a tubular H-frame, *Proceedings of the 17th IMAC*, Orlando, Florida, February 1999, pp. 1556–1562.
[6] C. Stavrinidis, J. Clinckemaillie, J. Dubois, New concepts for finite-element mass matrix formulation, *Proceedings of the 5th IMAC*, London, April 1987, pp. 1269–1277.
[7] C. Stavrinidis, J. Clinckemaillie, J. Dubois, New concepts for finite-element mass matrix formulation, *AIAA Journal* 27 (1989) 1249–1255.
[8] H. Ahmadian, M.I. Friswell, J.E. Mottershead, Minimization of the discretization error in mass and stiffness formulations by an inverse method, *International Journal for Numerical Methods in Engineering* 41 (1998) 371–387.
[9] H. Ahmadian, G.M.L. Gladwell, F. Ismail, Families of finite element models for updating procedures, *Proceedings of the 12th IMAC*, Honolulu, 1994, pp. 898–905.
[10] G.M.L. Gladwell, H. Ahmadian, Generic element matrices suitable for finite element model updating, *Mechanical Systems and Signal Processing* 9 (1995) 601–614.
[11] H. Ahmadian, G.M.L. Gladwell, F. Ismail, Parameter selection strategies in finite element model updating, *Journal of Vibration and Acoustics* 119 (1997) 37–45.
[12] M.J. Ratcliffe, N.A.J. Lieven, A generic element-based method for joint identification, *Mechanical Systems and Signal Processing* 14 (2000) 3–28.
[13] B. Titurus, M.I. Friswell, L. Starek, Damage detection using generic elements: part I model updating, *Computers and Structures* 81 (2003) 2273–2286.
[14] M.I. Friswell, J.E. Mottershead, M.G. Smart, Dynamic models of golf clubs, *Sports Engineering* 1 (1998) 41–50.
[15] D. Wu, S.S. Law, Model error correction from truncated modal flexibility sensitivity and generic parameters: part I—simulation, *Mechanical Systems and Signal Processing* 18 (2004) 1381–1399.
[16] D. Wu, S.S. Law, Model error correction from truncated modal flexibility sensitivity and generic parameters: part II—experimental verification, *Mechanical Systems and Signal Processing* 18 (2004) 1401–1419.
[17] S.S. Law, T.H.T. Chan, D. Wu, Super-element with semi-rigid joints in model updating, *Journal of Sound and Vibration* 239 (2001) 19–39.
[18] D. Wang, M.I. Friswell, P.E. Nikravesh, E.Y. Kuo, Damage identification in structural joints using generic joint elements, *Proceedings of the 17th IMAC*, Orlando, Florida, USA, February 1999, 792–798.
[19] R.I. Levin, T.P. Waters, N.A.J. Lieven, Required precision and valid methodologies for dynamic finite element model updating, *Journal of Vibration and Acoustics* 120 (1998) 733–741.
[20] A.N. Tikhonov, V.Y. Arsenin, *Solutions of Ill-Posed Problems*, Wiley, New York, 1977.
[21] P.C. Hansen, D.P. O'Leary, The use of the L-curve in the regularisation of discrete ill-posed problems, *SIAM Journal of Scientific Computation* 14 (1993) 1487–1503.